The Stability Analysis of T-S Fuzzy System with Constant Time-delay

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Abstract: To study the problems of stability analysis of non-linear system with time-delay, this paper focuses on the stability analysis of T-S fuzzy system with constant time-delay. By constructing a new type of Lyapunov-Krasovskii functional and employing integral inequality method based on Legendre orthogonal polynomials, a new stability criterion with less conservatism has proposed. Numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: T-S fuzzy system; Constant time-delay; Integral inequalities; Legendre orthogonal polynomials

1 Introduction

The stability analysis of non-linear system with time-delay, which is usual control plant in engineering application, have important research significance. It's well known that non-linear system with time-delay is very hard to study directly. Meanwhile, time-delay can also bring poor control performance, oscillation or even cause instability.

Considering that Tagaki-Sugeno (T-S) fuzzy system with time-delay can approximate a non-linear system with time-delay in arbitrary precision [1]. The stability analysis of non-linear system with time-delay can be transferred to those of T-S fuzzy system with time-delay. Therefore, T-S fuzzy system with time-delay has attracted great attention in the past decades and various approaches of stability analysis have been proposed.

As we all know, one of the most popular method of stability analysis is the Lyapunov-Krasovskii (L-K) functional method. However, L-K functional method can only obtain sufficient conditions, and the conservatism of stability criterion has connection with L-K function. Therefore, how to construct proper L-K function and which type of mathematic method to be applied for the derivatives of L-K functional is significant to reduce the conservatism of stability criterion.

Applying L-K functional to analyze the stability of T-S fuzzy system with time-delay, the derivative of L-K functional often contains integral term, such as: $-\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds$. Much effort has been devoted into the study of integral inequalities since tighter integral inequalities can reduce the conservatism of stability criterion. Considering single integral information of the state, the Jensen inequality and Wirtinger inequality was proposed [2] and [3], respectively. With multiple integral information of the state employed and more free matrices introduced, the stability criterion of T-S fuzzy system with time-delay becomes more and more relaxiable. The Bessel-Legendre inequality method based on Legendre orthogonal polynomial, which proposed in [4], has improved integral inequality method greatly.

However, there still exists somewhere to improve. The integral inequality should be more accurate by considering more orthogonal polynomials. Inspired by reference [5], which applies the integral inequalities based on Legendre orthogonal polynomials to analyze the stability of time-delay system, this paper further investigates the method proposed in [5] and extends it to the T-S fuzzy system with time-delay. By constructing a new type of L-K functional and considering more orthogonal polynomial terms, a stability criterion with less conservatism for the T-S fuzzy system with time-delay has proposed. To demonstrate the advantage of the proposed inequalities, some numerical examples are given.

2 Preliminaries

Considering a nonlinear system with time-delay that can be described by the following delayed T-S fuzzy system with r plant rules:

$$\begin{aligned} &R^i : If\ z_1(t)\ is\ F_1^i\ and \ldots z_g(t)\ is\ F_g^i,\ then \\ &\dot{x}(t) = A_{0i}x(t) + A_{di}x(t-h) \quad i = 1,2,\ldots,r \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state variable of T-S fuzzy system, A_{0i}, A_{di} are real constant matrices with appropriate dimensions. $F_1^i, \ldots, F_g^i (i = 1,2,\ldots,r)$ are fuzzy sets and the time-delay h of system is assumed to be known and constant.

By fuzzy blending, the overall fuzzy model is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} u_i\left(z(t)\right)\left(A_{0i}x(t) + A_{di}x(t-h)\right); \tag{2}$$

$$u_i\left(z(t)\right) = \frac{w_i\left(z(t)\right)}{\sum_{i=1}^{r} w_i\left(z(t)\right)};$$

$$w_i\left(z(t)\right) = \prod_{j=1}^{v} F_i^j\left(z(t)\right);$$

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The membership function $u_i\left(z(t)\right)$ satisfying

$$0 < u_i\left(z(t)\right) < 1; \quad \sum_{i=1}^{r} u_i\left(z(t)\right) = 1;$$

Lemma 1: For symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, where $S_{11} \in R^{r\times r}$. The following conditions are equivalent:

$$(1)\ S < 0;$$
$$(2)\ S_{11} < 0,\ S_{22} - S_{21}S_{11}^{-1}S_{12} < 0;$$
$$(3)\ S_{22} < 0,\ S_{11} - S_{12}S_{22}^{-1}S_{21} < 0;$$

Remark 1: Lemma 1 can transfer non-standard linear matrix inequality (LMI) in control problems to standard LMI form.

Lemma 2 [5]: Suppose that $x(t):[a,b] \to R^n$ is differential and continuous function. For given integers $m, N \in \mathbb{N}$ satisfying $m \ge N$, matrices $R \in S_n^+$ and $M_i \in R^{(m+2)n\times n}$, $i = 1,2,...,N$, the following inequality

$$-\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \le \vartheta^T \Phi \vartheta \tag{3}$$

holds, for any scalar function $p_k(s)$ satisfying:

$$\langle p_k, p_l \rangle \triangleq \int_a^b p_k(s)p_l(s)ds \begin{cases} = 0, k \ne l \in \mathbb{N} \\ \ne 0, k \ne l \in \mathbb{N} \end{cases} \tag{4}$$

where

$$\vartheta = \begin{bmatrix} x(b) \\ x(a) \\ \frac{1}{s_1}\Omega_{o^1} \\ \vdots \\ \frac{1}{s_m}\Omega_{o^m} \end{bmatrix};$$

$$S_i = \int_a^b \int_{u_1}^b \cdots \int_{u_{i-1}}^b du_i \ldots du_2 du_1;$$

$$\Omega_{o^m} = \int_a^b \int_{u_1}^b \cdots \int_{u_{m-1}}^b x(u_m)du_m \cdots du_2 du_1;$$

$$\Phi = \sum_{i=0}^{N} \langle p_i, p_i \rangle M_i R^{-1} M_i^T + \sum_{i=0}^{N} sym\{M_i \Pi_i\};$$

where $sym\{M_i\Pi_i\} = M_i\Pi_i + \left(M_i\Pi_i\right)^T$ and Π_i satisfy the following equation:

$$\int_a^b p_i(s)\dot{x}(s)ds = \Pi_i \vartheta, \quad i = 1,2,...,N \tag{5}$$

The Legendre polynomials satisfy orthogonal property(4). The first four Legendre polynomials are considered, namely $N = 3$

$$L_0(s) = 1;$$
$$L_1(s) = -1 + 2\left(\frac{S_a}{b_a}\right);$$
$$L_2(s) = 1 - 6\left(\frac{S_a}{b_a}\right) + 6\left(\frac{S_a}{b_a}\right)^2;$$
$$L_3(s) = -1 + 12\left(\frac{S_a}{b_a}\right) - 30\left(\frac{S_a}{b_a}\right)^2 + 20\left(\frac{S_a}{b_a}\right)^3;$$

where $S_a = s - a$ and $b_a = b - a$. So the integral inequality based on Legendre orthogonal polynomials is derived with $m = N = 3$.

Corollary 1 [5]: Suppose that $x(t):[a,b] \to R^n$ is differential and continuous function. For given integers $m, N \in \mathbb{N}$ satisfying $m \ge N$, matrices $R \in S_n^+$ and $M_i \in R^{(m+2)n\times n}$, $i = 1,2,...,N$, the following inequality

$$-\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \le \varsigma^T \Theta \varsigma \tag{6}$$

Where

$$\varsigma = col\left\{x(b), x(a), \frac{1}{b_a}\Omega_{o^1}, \frac{2}{b_a}\Omega_{o^2}, \frac{6}{b_a}\Omega_{o^3}\right\}$$

$$\Theta = \sum_{i=0}^{3} \frac{b_a}{(2i+1)} M_i R^{-1} M_i^T + \sum_{i=0}^{3} sym\{M_i \Pi_i\} \tag{7}$$

where

$$\Pi_0 = e_1 - e_2;$$
$$\Pi_1 = e_1 + e_2 - 2e_3;$$
$$\Pi_2 = e_1 - e_2 + 6e_3 - 6e_4;$$
$$\Pi_3 = e_1 + e_2 - 12e_3 + 30e_4 - 20e_5;$$
$$e_i = [0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(5-i)n}], \quad i = 1,2,...,5$$

3 Stability Criterion

Considering T-S fuzzy system with constant delay (1) and applying Corollary 1, a stability criteria is derived in Theorem 1 with a new type of L-K functional.

Theorem 1: For given constant time-delay

$h \in [h_{\min}, h_{\max}]$, if there exist symmetric positive definite matrix $P \in R^{4n\times 4n}, Q, R \in R^{n\times n}$ and matrices $M_i \in R^{5n\times n}$, $i = 0,1,2,3$, such that the following linear matrix inequalities:

$$\begin{bmatrix} \Psi_i & hM_0 & hM_1 & hM_2 & hM_3 \\ * & -hR & 0 & 0 & 0 \\ * & * & -3hR & 0 & 0 \\ * & * & * & -5hR & 0 \\ * & * & * & * & -7hR \end{bmatrix} < 0,\ i = 1, 2, \ldots, r \quad (8)$$

holds, then T-S fuzzy system with constant time-delay (1) is asymptotically stable, where

$$\Psi_i = \Psi_{1i} + \sum_{k=0}^{3} sym\{M_k \Pi_k\};$$
$$\Psi_{1i} = sym\{\Gamma_{2i}^T P \Gamma_1\} + e_1^T Q e_1 - e_2^T Q e_2 + h e_{0i}^T R e_{0i};$$

$$\Gamma_1 = \begin{bmatrix} e_1 \\ he_3 \\ \frac{h^2}{2} e_4 \\ \frac{h^3}{6} e_5 \end{bmatrix}, \qquad \Gamma_{2i} = \begin{bmatrix} e_{0i} \\ e_1 - e_2 \\ h(e_1 - e_3) \\ \frac{h^2}{2}(e_1 - e_4) \end{bmatrix}, i = 1, 2, \ldots, r$$

$$\Pi_0 = e_1 - e_2;$$
$$\Pi_1 = e_1 + e_2 - 2e_3;$$
$$\Pi_2 = e_1 - e_2 + 6e_3 - 6e_4;$$
$$\Pi_3 = e_1 + e_2 - 12e_3 + 30e_4 - 20e_5;$$

$$e_i = [0_{n\times(i-1)n} \ \ I_n \ \ 0_{n\times(5-i)n}], \quad i = 1, 2, \ldots, 5$$
$$e_{0i} = A_{0i} e_1 + A_{di} e_2, \quad i = 1, 2, \ldots, 5$$

Proof: Constructing the following L-K functional:

$$V(t) = \eta^T(t) P \eta(t) + \int_{t-h}^{t} x^T(s) Q x(s) ds + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta \quad (9)$$

where

$$\eta(t) = \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)ds \\ \int_{-h}^{0}\int_{t+u}^{t} x(s)dsdu \\ \int_{-h}^{0}\int_{l}^{0}\int_{t+u}^{t} x(s)dsdudl \end{bmatrix};$$

the derivative of (9) along the trajectory of system is computed as

$$\begin{aligned} \dot{V}(t) = {} & \dot{\eta}^T P \eta + \eta^T P \dot{\eta} + x^T(t) Q x(t) \\ & - x^T(t-h) Q x(t-h) + h\dot{x}^T(t) R \dot{x}(t) \\ & - \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds \end{aligned} \quad (10)$$

where

$$\dot{\eta}(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-h) \\ hx(t) - \int_{t-h}^{t} x(s)ds \\ \frac{h^2}{2} x(t) - \int_{-h}^{0}\int_{t+u}^{t} x(s)dsdu \end{bmatrix};$$

Define:

$$\vartheta = \begin{bmatrix} x(t) \\ x(t-h) \\ \frac{1}{h}\int_{t-h}^{t} x(s)ds \\ \frac{2}{h^2}\int_{-h}^{0}\int_{t+u}^{t} x(s)dsdu \\ \frac{6}{h^3}\int_{-h}^{0}\int_{l}^{0}\int_{t+u}^{t} x(s)dsdudl \end{bmatrix};$$

We can see that

$$\dot{\eta}(t) = \Gamma_{2i}\vartheta(t),\ i = 1, 2, \ldots, r;$$
$$\eta(t) = \Gamma_1 \vartheta(t); \qquad x(t) = e_1 \vartheta(t);$$
$$x(t-h) = e_2 \vartheta(t); \quad \dot{x}(t) = e_{0i}\vartheta(t);$$

so that

$$\begin{aligned} \dot{V}(t) = {} & \vartheta^T \Gamma_{2i}^T P \Gamma_1 \vartheta + \vartheta^T \Gamma_1^T P \Gamma_{2i} \vartheta + \vartheta^T e_1^T Q e_1 \vartheta \\ & - \vartheta^T e_2^T Q e_1 \vartheta + h \vartheta^T e_{0i}^T R e_{0i} \vartheta \\ & - \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds \\ = {} & \vartheta^T \psi_{1i} \vartheta - \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds \end{aligned}$$

Applying Corollary 1, we get

$$\dot{V}(t) \le \vartheta^T \psi_{1i} \vartheta + \vartheta^T \Phi \vartheta$$
$$= \vartheta^T \left\{ \psi_i + \sum_{k=0}^{3} \frac{h}{(2k+1)} M_k R^{-1} M_k^T \right\}$$

where $\Psi_i = \Psi_{1i} + \sum_{k=0}^{3} sym\{M_k \Pi_k\}$.

If the following inequality:

$$\psi_i + \sum_{k=0}^{3} \frac{h}{(2k+1)} M_k R^{-1} M_k^T < 0$$

we can see that it's equivalent to (8), according to the Lemma 1. Then

$$\dot{V}(t) \le \vartheta^T \left\{ \psi_i + \sum_{k=0}^{3} \frac{h}{(2k+1)} M_k R^{-1} M_k^T \right\} < 0$$

So that T-S fuzzy system with constant time-delay (1) is asymptotically stable. This completes the proof.

4 Numerical example

In this section, some numerical examples are given to show the effectiveness of the proposed method.

Example 1 [9]: Considering T-S fuzzy system with constant time-delay (1) with r = 2. The membership function of rule 1and rule 2 as follow:

$$u_1(z_1(t)) = \frac{1}{1+\exp(-2z_1(t))};$$
$$u_2(z_2(t)) = 1 - u_1(z_1(t));$$

where

$$A_{01} = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix};$$
$$A_{02} = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix};$$

Applying Theorem 1, we can get the MAUB of the system in example 1 is 2.01926. The conservatism is compared among other stability criteria, which are listed in Table I. we can see that Theorem 1 is less conservative.

Table I Upper Bounds on h obtained for example 1

Paper	MAUBs	Paper	MAUBs
Chen *et al*.[6]	1.5974	Idrissi (N=1)[9]	1.6341
Wu (Coro. 1)[7]	1.5974	An *et al.*(N=1)[10]	1.8980
Peng (Coro. 2)[8]	1.5974	An *et al.*(N=2)[10]	1.9560
Peng (Coro. 2)[8]	1.6341	Theorem 1	2.0193

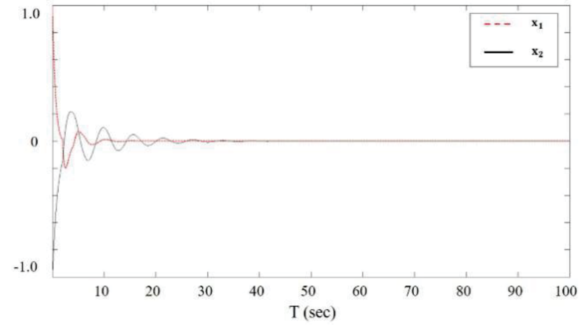

Figure 1 State responses of the open-loop system for example 1.

We can see that the system in example 1 with $h = 2.01926$ is asymptotically stable from **Figure 1**.

Example 2 [11]: Considering T-S fuzzy system with constant time-delay (1) with r = 2. The membership function of rule 1and rule 2 as follow:

$$u_1(z_1(t)) = \frac{1}{1+\exp(-2z_1(t))};$$
$$u_2(z_2(t)) = 1 - u_1(z_1(t));$$

where

$$A_{01} = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix};$$
$$A_{02} = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix};$$

Applying Theorem 1, we can get the MAUB of the system in example 2 is 3.15689. It's shown in Table II that the conservatism of Theorem 1 is much less.

Table II Upper Bounds on h obtained for example 2

Paper	MAUBs	Paper	MAUBs
Fang *et al.*[11]	2.6500	Zhang *et al.*[13]	3.0900
An *et al.*[12]	3.0230	Theorem 1	3.1569

5 Conclusions

In this paper, a new type of L-K functional has constructed and integral inequality method based on Legendre orthogonal polynomials has employed. The stability criterion derived in this paper is less conservative than existing results. It's worth pointing out that the proposed criterion is only sufficient condition of stable system. The maximum allowable delay obtained from solving stability criterion may reach its analytic maximum value, if the number of orthogonal polynomial terms tends to infinity. In another word, the greater the number of orthogonal polynomial terms, the less conservative the stability criterion is. To reduce conservatism, the integral inequality method proposed in this paper can be combined with delay-partitioning technique. Meanwhile, the stability criterion can be applied in the state-feedback controller design of T-S fuzzy system with time-delay.

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